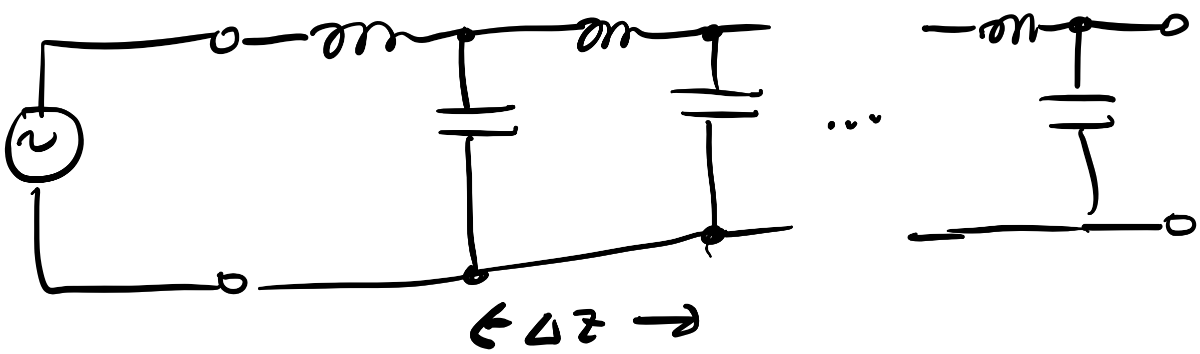
PHYS 331 - Oct. 16, 2023

Last Time



Found $V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z}$

$$\beta = \omega\sqrt{L_l C_l}$$

Useful results:

(#) $$\frac{\partial V}{\partial z} = -j\omega L_l I$$

(*) $$\frac{\partial V}{\partial z} = -j\beta V_+ e^{-j\beta z} + j\beta V_- e^{j\beta z}$$

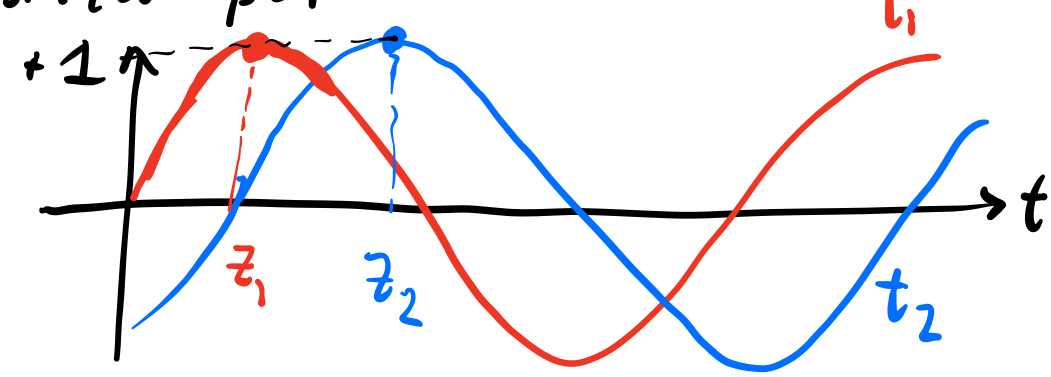
Full time-dep. voltage on trans. line is:

$$v(z,t) = V(z) e^{j\omega t}$$

$$= \left(V_+ e^{-j\beta z} + V_- e^{j\beta z} \right) e^{j\omega t}$$

$$= V_+ e^{j(\omega t - \beta z)} + V_- e^{j(\omega t + \beta z)}$$

consider $\sin(\omega t - \beta z)$



Require $\sin(\omega t_1 - \beta z_1) = \sin(\omega t_2 - \beta z_2)$

$$\Downarrow$$

$$\omega t_1 - \beta z_1 = \omega t_2 - \beta z_2$$

$$\omega \underbrace{(t_2 - t_1)}_{\Delta t} = \beta \underbrace{(z_2 - z_1)}_{\Delta z}$$

$$\therefore \frac{\omega}{\beta} = \underbrace{\frac{\Delta z}{\Delta t}}_{S}$$

S the speed of the signal.

$$S = \frac{\omega}{\beta} \qquad \text{know } \beta = \omega\sqrt{L_l C_l}$$

$$\therefore S = \frac{\cancel{\omega}\,1}{\cancel{\omega}\sqrt{L_l C_l}} \qquad \boxed{\therefore S = \frac{1}{\sqrt{L_l C_l}}}$$

For parallel conductors of radius a, we had

$$L_\ell = \frac{\mu_0 \ln\left(\frac{d}{a}\right)}{\pi}$$

$$C_\ell = \frac{\pi \varepsilon_0}{\ln\left(\frac{d}{a}\right)}$$

$$\therefore L_\ell C_\ell = \mu_0 \varepsilon_0 = \frac{1}{c^2}$$

$\therefore$ signal speed in transmission line is

$$s = \frac{1}{\sqrt{L_\ell C_\ell}} = c$$ the speed of light.

If our parallel wires were in a medium with relative permittivity ε_r & permeability μ_r, then we would find:

$$S = \frac{c}{\sqrt{\varepsilon_r \mu_r}} = \frac{c}{n}$$ where n is the refractive index.

If we analyze the $\omega t + \beta z$ term, we would find the same result w/ a negative sign. $S = -\frac{1}{\sqrt{L_\ell C_\ell}} = -c$

$e^{j(\omega t - \beta z)}$: forward travelling wave

$e^{j(\omega t + \beta z)}$: backwards travelling wave. Can only be due to reflections from the end of the trans. line.

What is the physical interpretation of β?

$$s = \frac{\omega}{\beta} = f\lambda = \frac{(2\pi f)\lambda}{2\pi}$$

$$\therefore \frac{\cancel{\omega}}{\beta} = \cancel{\omega}\frac{\lambda}{2\pi}$$

$$\boxed{\beta = \frac{2\pi}{\lambda} \quad \text{called propagation const. or wave number}}$$

Return to useful results ④ & ⑥

- two expressions for dV/dz which must be equal to one another.

$$-\cancel{j}\beta V_+ e^{-j\beta z} + \cancel{j}\beta V_- e^{j\beta z} = -\cancel{j}\omega L_\ell I$$

solve for $I(z)$.

$$I(z) = \frac{\cancel{+}\beta\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]}{\cancel{+}\omega L_\ell}$$

$$= \frac{\beta}{\omega L_\ell}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]$$

$$\beta = \omega\sqrt{L_\ell C_\ell}$$

$$\therefore \quad \frac{\beta}{\omega L_\ell} = \frac{\cancel{\omega}\sqrt{L_\ell C_\ell}}{\cancel{\omega} L_\ell} = \sqrt{\frac{C_\ell}{L_\ell}}$$

$$\therefore \quad I(z) = \sqrt{\frac{C_\ell}{L_\ell}}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right]$$

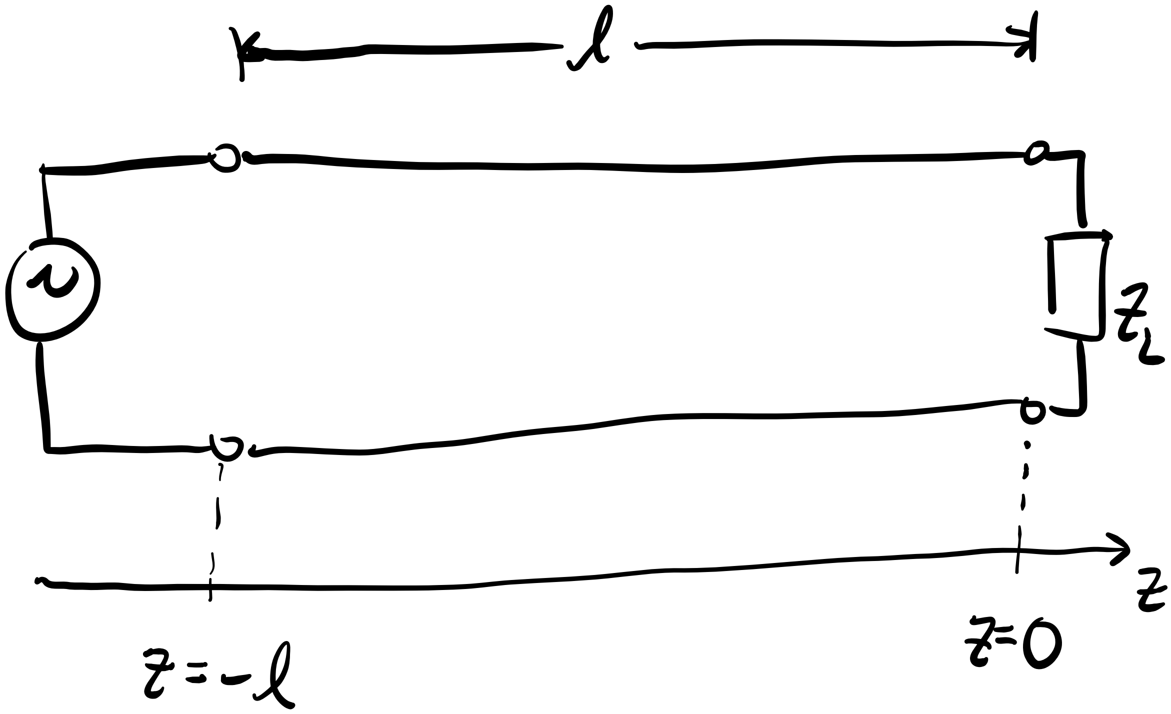
the factor $\sqrt{\frac{C_\ell}{L_\ell}}$ must have units of Ω!

Define $\sqrt{\frac{L_\ell}{C_\ell}}$ as the "characteristic impedance" Z_0 of the transmission line. $Z_0 = \sqrt{\frac{L_\ell}{C_\ell}}$

Summary: For a trans. line driven by a harmonic signal, the volt. & current amplitudes along length of trans. line are given by:

$$\boxed{\begin{aligned} V(z) &= V_+ e^{-j\beta z} + V_- e^{j\beta z} \\ I(z) &= \frac{1}{Z_0}\left[V_+ e^{-j\beta z} - V_- e^{j\beta z}\right] \end{aligned}}$$

Consider a trans. line of length ℓ terminated by a "load" impedance Z_L at its end.



With the chose coord. system, the voltage & current amp. at the pos. of Z_L ($z=0$) are given by:

$$V(0) = V_+ + V_-$$

$$I(0) = \frac{1}{Z_0}\left[V_+ - V_-\right]$$

The ration $\frac{V(0)}{I(0)}$ must be equal to the load impedance Z_L at $z=0$

$$\frac{V(0)}{I(0)} = Z_L = \underbrace{Z_0\left[\frac{V_+ + V_-}{V_+ - V_-}\right]}$$

rearrange to express V_- in terms of V_+, Z_0, Z_L.

$$\Downarrow$$

Find $\boxed{V_- = V_+\underbrace{\left(\frac{Z_L - Z_0}{Z_L + Z_0}\right)}}$

reflection coefficient.

amp. of backwards travelling wave

amp. of forward travelling wave

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

transmission line reflection coefficient.

$\hookrightarrow$ $V_- = \Gamma V_+$

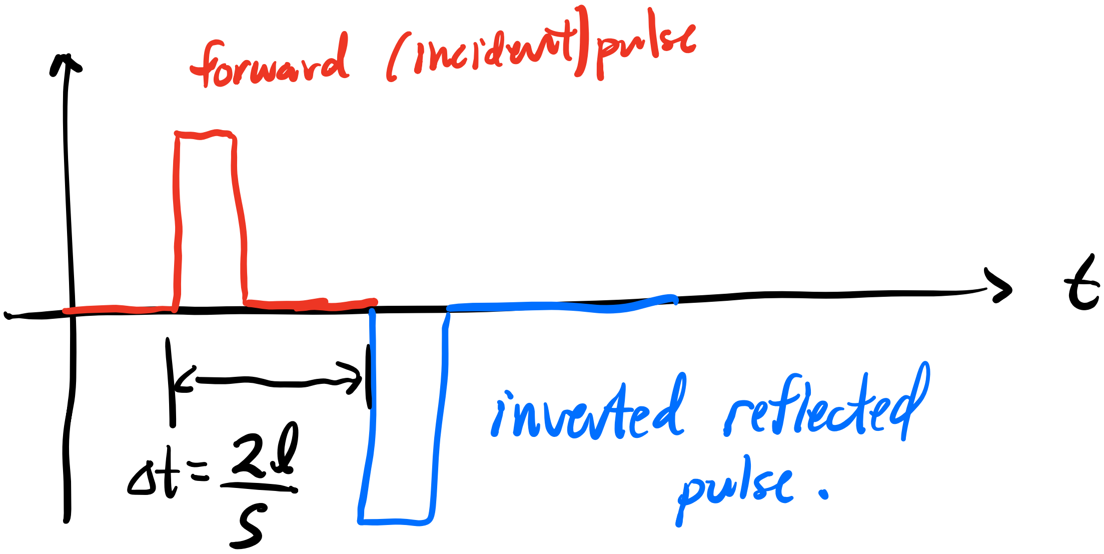
Special cases:

① $Z_L = 0$ (short circuit)

$$\Gamma = \frac{0 - Z_0}{0 + Z_0} = -1$$

signal is inverted upon reflection (phase shift of 180°).

$|\Gamma| = 1 \Rightarrow$ perfect reflection



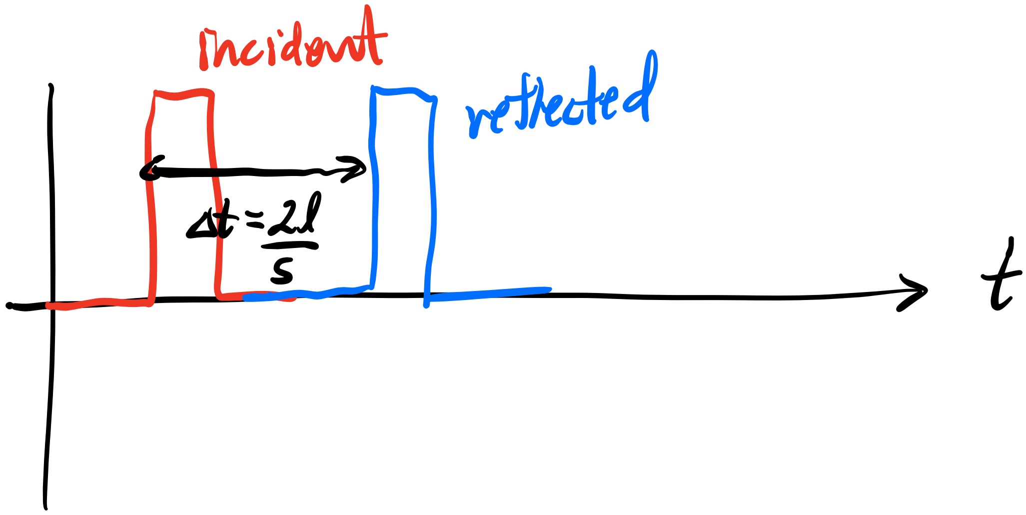
time for pulse to travel to end of trans. line & back again.

② $Z_L \to \infty$ (open circuit)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \approx \frac{Z_L}{Z_L} = +1.$$

another perfect reflection

No inversion.



③ $Z_L = Z_0$ (impedance matching)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_0 - Z_0}{2Z_0} = 0$$

no reflection.
All signal absorbed by load impedance Z_L.

$$\lambda f = c$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8\ \mathrm{m/s}}{60\ \frac{1}{s}} = 0.5\times10^7\ \mathrm{m}$$

$$= 50\times10^6\ \mathrm{m}$$